Approximating All-Pair Bounded-Leg Shortest Path and APSP-AF in Truly-Subcubic Time

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Outline

1. Introduction
   - Motivation
   - Our results
2. The algorithm
   - 1. Exact product for small distances
   - 2. Approximate product for arbitrary distances
   - 3. Main procedure
3. (Sketch of) a faster algorithm
4. Conclusions and open problems

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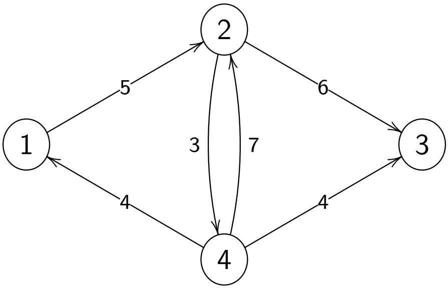

A graph G.

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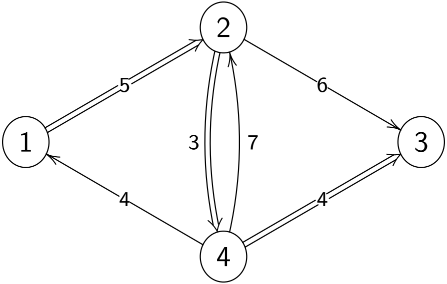

$Query(s = 1, t = 3, L = 5)$

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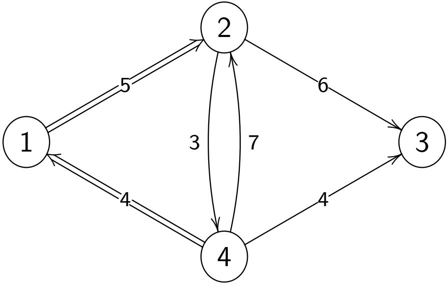

$Query(s = 4, t = 2, L = 6)$

APSP-AF

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- A generalization of bounded-leg shortest path.

Our results

- A simple algorithm for $(1+\epsilon)$-approximating APSP-AF in $\tilde{O}(n^{\frac{3+\omega}{2}}\epsilon^{-2}\log W)$ preprocessing time[1] and $O(\log\frac{\log(nW)}{\epsilon})$ query time
  - where W is the maximum edge length,
  - and $\omega < 2.373$ is the matrix-multiplication exponent.

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- An algorithm in $\tilde{O}(n^{\frac{3+\omega}{2}}\epsilon^{-3/2}\log W)$ preprocessing time
- This is the first truly-subcubic[2] time algorithm for such problems.

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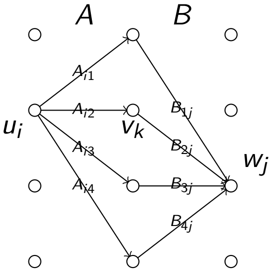
Various Matrix Products

- Distance product: $(A \star B)_{ij} = \min_k\{A_{ik} + B_{kj}\}$.



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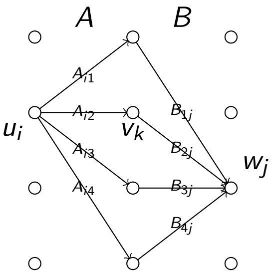

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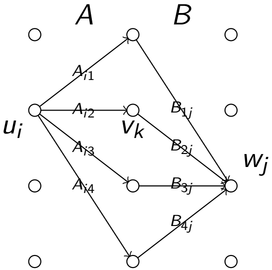

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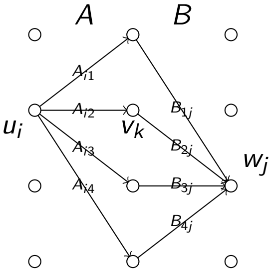

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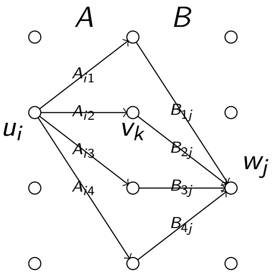
  - $A \circledvee B$ can be computed in $O(n^{\frac{3+\omega}{2}})$ time [Duan and Pettie, 2009].



Query Time

- We compute a matrix whose entries A_{ij} are sets of (d, f) pairs.
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- Query time $O(\log |A_{st}|)$, and $|A_{st}| = O(\frac{\log(nW)}{\epsilon})$.

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- For a df-matrix A and a flow f, define $A(f)$ be the matrix satisfying $A(f)_{ij} = \min\{d : \exists(d, f') \in A_{ij}, f' \geq f\}$.
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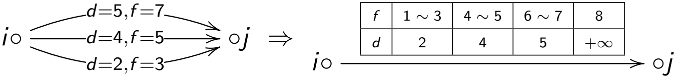

f	$1 \sim 3$	$4 \sim 5$	$6 \sim 7$	8
d	2	4	5	$+\infty$

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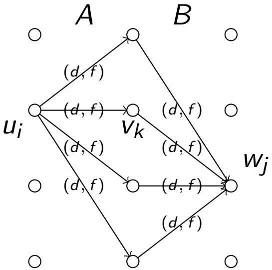

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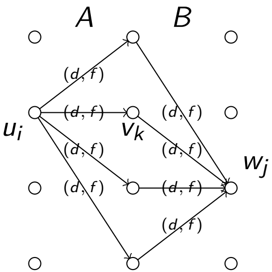

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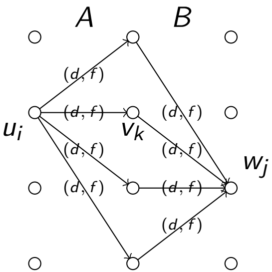

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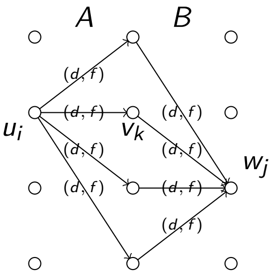
- This takes $O(n^{\frac{3+\omega}{2}} R^2)$ time.



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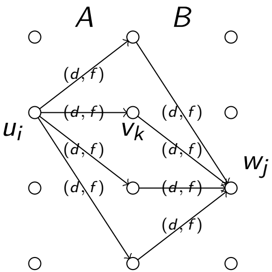

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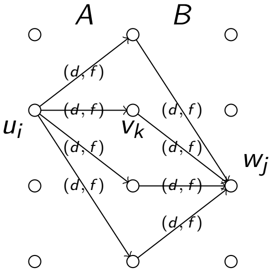

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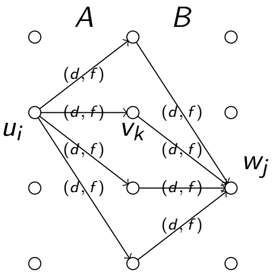

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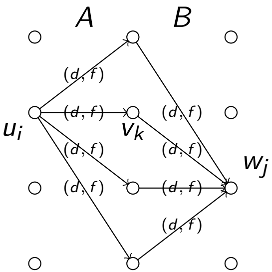

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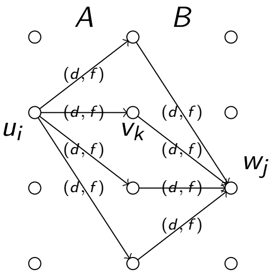

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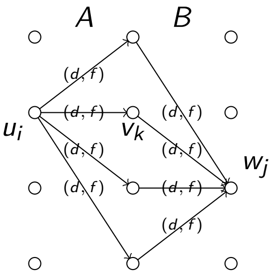

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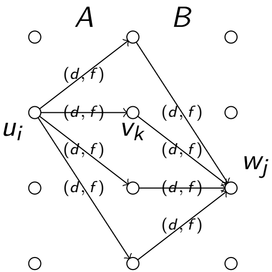
  - Recall that $\otimes$ is max-min product.
- Let's define $A_{ij}^{(d)} = \max\{f : (d', f) \in A_{ij}, d' \le d\}$.
- $O(R^2)$ max-min products suffice to compute the df-matrix C!
- $C_{ij} = \{(\max_{d_1+d_2=d}(A^{(d_1)} \otimes B^{(d_2)})_{ij}, f) : 1 \le d \le 2R\}$



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- time complexity: $O(n^{\frac{3+\omega}{2}} R^2 \log M)$.
  - $O(\log M)$ exact products in which $d \leq R$.

A lemma

Lemma ([Zwick, 1998])

Let A, B be two matrices with entries in $\{0, 1, \ldots, M, +\infty\}$, $C = A \star B$.

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Define C' as:

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then for any i, j, $C_{ij} \le C'_{ij} \le (1 + \frac{4}{R})C_{ij}$.

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- (This $\star$ is the previous exact product.)

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- It can be proved by induction that $\tilde{D}^{(\lceil \log_2 n \rceil)}$ is an $(1 + \frac{4}{R})^{\lceil \log_2 n \rceil}$-approximation of APSP-AF distances.

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- Let $\tilde{D}^{(0)} = G$, $\tilde{D}^{(i)}$ be an approximation of $\tilde{D}^{(i-1)} \star \tilde{D}^{(i-1)}$.
- It can be proved by induction that $\tilde{D}^{(\lceil \log_2 n \rceil)}$ is an $(1 + \frac{4}{R})^{\lceil \log_2 n \rceil}$-approximation of APSP-AF distances.
- We set R the smallest power of 2 greater than $4\lceil \log_2 n \rceil / \ln(1 + \epsilon)$, and we're done.

3. Main procedure

- We're ready to approximate the APSP-AF problem.
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- We set R the smallest power of 2 greater than $4\lceil \log_2 n \rceil / \ln(1 + \epsilon)$, and we're done.
- Time complexity: $O(n^{\frac{3+\omega}{2}} R^2 \log M \log n) = \tilde{O}(n^{\frac{3+\omega}{2}} \epsilon^{-2} \log M)$.

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- Open problems
  - Reducing dependency on n(*i.e.* $\frac{3+\omega}{2}$) requires faster max-min product. But can APSP-AF be done in $\tilde{O}(n^{\frac{3+\omega}{2}}\epsilon^{-1}\log W)$?

Thank you!

Questions are welcome!

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