Faster Construction Algorithms for Distance Sensitivity Oracles

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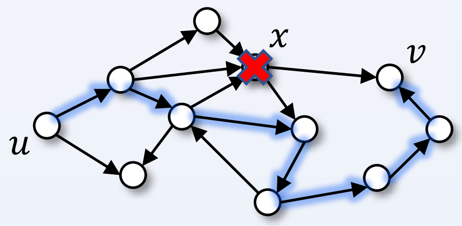
Distance Sensitivity Oracles (DSOs):

- Input: a directed graph $G=(V,E)$
- On each query (u,v,x), find the length of the shortest path from u to v that avoids x
- Notation: $||uv \diamond x||$ is the length of the sought path



Assumption: edges are unweighted.

- $\Longrightarrow$ All-pairs shortest paths can be computed in $O(n^{2.5286})$ time [Zwick], using fast matrix multiplication

Idea 1: Bootstrapping DSOs

- DSO with preproc. time P and query time $Q \Longrightarrow$ DSO with preproc. time $P+\tilde{O}(n^2)\cdot Q$, query time $O(1)$

Proof sketch:

Preprocess a DSO using [Bernstein-Karger].

The preprocessing algorithm of [Bernstein-Karger] can be summarized as three steps:

1. Compute $\tilde{O}(n^2)$ DSO queries $\{(u_i,v_i,x_i)\}_{i\le\tilde{O}(n^2)}$
2. Answer the queries (somehow) in $\tilde{O}(mn)$ time
3. Construct the DSO (somehow) from these answers.

To prove our bootstrapping theorem, we can simply replace step 2 by "answering the queries using the slower DSO", which takes $P+\tilde{O}(n^2)\cdot Q$ time.

Idea 2: Hitting Sets

An **r-truncated DSO** is a DSO that on a query (u,v,x), only needs to return $\min\{||uv\diamond x||, r\}$ instead of $||uv\diamond x||$.

- $(+\infty)$-truncated DSO = (normal) DSO

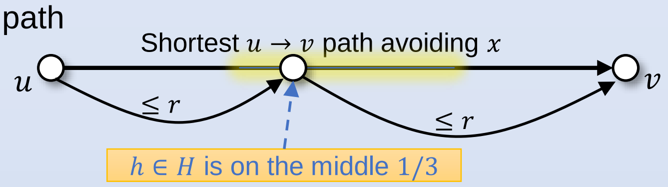
We can use hitting sets to transform an r-truncated DSO with query time $O(1)$ into a $(3/2)r$-truncated DSO with query time $\tilde{O}(n/r)$.

Let $H \leftarrow$ a random sample of $\tilde{O}(n/r)$ vertices

Query algorithm for the $(3/2)r$-truncated DSO:

- On input (u,v,x), if $||uv\diamond x|| \le r$, return $||uv\diamond x||$
- Otherwise, return $\min_{h\in H}\left\{||uh\diamond x||_r + ||hv\diamond x||_r\right\}$ using the old r-truncated DSO
- Here $||uv\diamond x||_r := \min\{||uv\diamond x||, r\}$

Correctness: assuming $r \le ||uv\diamond x|| \le (3/2)r$, there is at least one vertex in H that hits the middle $1/3$ part of the sought path



Paper	Preproc. time	Query Time
[Demetrescu et al. 02]	$\tilde{O}(mn^2)$	$O(1)$
[Bernstein & Karger 09]	$\tilde{O}(mn)$	$O(1)$
[Weimann & Yuster 10]	$\tilde{O}(n^{1-\alpha+\omega})$	$\tilde{O}(n^{1+\alpha})$
[Grandoni & Williams 12]	$\tilde{O}(n^{\omega+1/2} + n^{\omega+\alpha(4-\omega)})$	$\tilde{O}(n^{1-\alpha})$
[Chechik & Cohen 20]	$O(n^{2.873})$	$\mathrm{polylog}(n)$
[Ren 20]	$O(n^{2.7233})$	$O(1)$
[Gu & Ren 21]	$O(n^{2.5794})$	$O(1)$

$\omega < 2.373$ is matrix multiplication exponent, $\alpha\in(0,1)$ is an arbitrary parameter

Idea 3: An r-Truncated DSO for Small r

Theorem: an r-truncated DSO can be constructed in time

$$r^2\cdot \mathrm{MM}(n,n/r,n/r)\cdot n^{o(1)}.$$

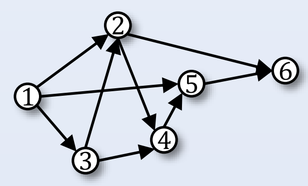
Here, $\mathrm{MM}(a,b,c)$ denotes the time complexity of multiplying an $a\times b$ matrix and a $b\times c$ matrix.

Adjoint of Symbolic Adjacency Matrix:

Symbolic adjacency matrix ($z_{i,j}$ are random numbers*)

* Actually, $z_{i,j}$ are analyzed as symbols that are in the end substituted by random numbers, hence the name "symbolic adjacency matrix".

$$\mathrm{SA}(G)_{i,j} = \begin{cases} 1 & i=j \\ z_{i,j}x & (i\to j)\in G \\ 0 & \text{otherwise}\end{cases}$$



$$\mathrm{SA}(G) = \begin{bmatrix} 1 & 38x & 22x & 0 & 81x & 0 \\ 0 & 1 & 0 & 28x & 0 & 37x \\ 0 & 65x & 1 & 70x & 0 & 0 \\ 0 & 0 & 0 & 1 & 19x & 0 \\ 0 & 0 & 0 & 0 & 1 & 60x \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Adjoint matrix: $\mathrm{adj}(A) = \det(A)\cdot A^{-1}$

Theorem: w.h.p. over the randomness of $\{z_{i,j}\}$, the lowest degree of x in $\mathrm{adj}(\mathrm{SA}(G))_{u,v}$ equals the $u\to v$ distance.

Ex: $\mathrm{adj}(\mathrm{SA}(G))_{3,6} = 2\,074\,800x^4 - 79\,800x^3 + 2\,405x^2$, hence the distance from 3 to 6 is 2.

Handling A Vertex Failure:

Trick: we can modulo every polynomial by x^r !!

Preprocessing: simply compute $\mathrm{adj}(\mathrm{SA}(G)) \bmod x^r$

- Time complexity: $r^2\cdot\mathrm{MM}(n,n/r,n/r)\cdot n^{o(1)}$

Handling failure of vertex x: $\mathrm{SA}(G-x) = \mathrm{SA}(G)+F_x$, where F_x is a certain rank-1 matrix uv^{T} corresponding to x

Sherman-Morrison-Woodbury formula (maintain $\mathrm{SA}(G)^{-1}$):

$$(A+uv^{\mathrm{T}})^{-1} = A^{-1} - (1+v^{\mathrm{T}}A^{-1}u)(A^{-1}uv^{\mathrm{T}}A^{-1}).$$

Query time: (turns out to be) $\tilde{O}(r)$

Idea 4: Unique Shortest Paths in $\tilde{O}(n^{2.5286})$ Time (omitted)

Putting It Together

r-truncated DSO with $\tilde{O}(r)$ query time $\xrightarrow{\text{Bootstrapping}}$ r-truncated DSO with $O(1)$ query time $\xrightarrow{\text{Hitting Sets}}$

$(3/2)r$-truncated DSO with $\tilde{O}(n/r)$ query time $\xrightarrow{\text{Bootstrapping}}$ $(3/2)r$-truncated DSO with $O(1)$ query time

$\xrightarrow{\text{Hitting Sets}}$ $(9/4)r$-truncated DSO with $\tilde{O}(n/r)$ query time $\cdots\Longrightarrow$ n-truncated DSO with $O(1)$ query time

- Preprocessing time: $r^2\cdot\mathrm{MM}(n,n/r,n/r)+n^3/r \xrightarrow{r:=n^{0.4206}} O(n^{2.5794})$
- Query time: $O(1)$

Note: bootstrapping also works for r-truncated DSOs

Hanlin Ren. Improved Distance Sensitivity Oracles with Subcubic Preprocessing Time. ESA'20.

Yong Gu and Hanlin Ren. Constructing a Distance Sensitivity Oracle in $O(n^{2.5794}M)$ Time. Submitted.